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Some subordination criteria concerning Sălăgean operator

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Abstract

Applying Sălăgean operator, for the class $\mathcal{A}$ of analytic functions $f(z)$ in the open unit disk $\mathbb{U}$ which are normalized by $f(0)=f'(0)-1=0$, the generalization of an analytic function to discuss the starlikeness is considered. Furthermore, from the subordination criteria for Janowski functions generalized by some complex parameters, some interesting subordination criteria for $f(z)\in\mathcal{A}$ are given.

1 Introduction, definition and preliminaries

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form:

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U}=\{z:z\in\mathbb{C}\quad\text{and}\quad|z|<1\}$.

Furthermore, let $\mathcal{P}$ denote the class of functions $p(z)$ of the form:

$$p(z)=1+\sum_{n=1}^{\infty}p_nz^n \tag{1.2}$$

which are analytic in $\mathbb{U}$. If $p(z)\in\mathcal{P}$ satisfies $\operatorname{Re}(p(z))>0\quad(z\in\mathbb{U})$, then we say that $p(z)$ is the Carathéodory function (cf. [1]).

A function $f(z)\in\mathcal{A}$ is said to be starlike of order α in $\mathbb{U}$ if it satisfies

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>\alpha\qquad(z\in\mathbb{U}) \tag{1.3}$$

for some α $(0\leqq\alpha<1)$. We denote by $\mathcal{S}^*(\alpha)$ the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which are starlike of order α in $\mathbb{U}$.

Similarly, if $f(z)\in\mathcal{A}$ satisfies the following inequality

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha\qquad(z\in\mathbb{U}) \tag{1.4}$$

for some α $(0 \leqq \alpha < 1)$, then $f(z)$ is said to be convex of order α in $\mathbb{U}$. We denote by $\mathcal{K}(\alpha)$ the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which are convex of order α in $\mathbb{U}$. As usual, in the present investigation, we write

$$\mathcal{S}^*(0) \equiv \mathcal{S}^* \quad \text{and} \quad \mathcal{K}(0) \equiv \mathcal{K}.$$

The classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ were introduced by Robertson [7].

By the familiar principle of differential subordination between analytic functions $f(z)$ and $g(z)$ in $\mathbb{U}$, we say that $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$ if there exists an analytic function $w(z)$ satisfying the following conditions:

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \qquad (z \in \mathbb{U}),$$

such that

$$f(z) = g\big(w(z)\big) \qquad (z \in \mathbb{U}).$$

We denote this subordination by

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}).$$

In particular, if $g(z)$ is univalent in $\mathbb{U}$, then it is known that

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}) \quad \Longleftrightarrow \quad f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

For $p(z) \in \mathcal{P}$, we introduce the following function

$$p(z) = \frac{1+Az}{1+Bz} \qquad (-1 \leqq B < A \leqq 1) \tag{1.5}$$

which has been investigated by Janowski [3]. Thus, the function $p(z)$ given by (1.5) is said to be the Janowski function.
Here, for some A and B $(-1 < B < A \leqq 1)$, the function $p(z)$ given by (1.5) is analytic and univalent in $\mathbb{U}$ and $p(z)$ maps the open unit disk $\mathbb{U}$ onto the open disk given by

$$\left| p(z) - \frac{1-AB}{1-B^2} \right| < \frac{A-B}{1-B^2}.$$

Thus, it is clear that

$$\operatorname{Re}\big(p(z)\big) > \frac{1-A}{1-B} \geqq 0 \qquad (z \in \mathbb{U}). \tag{1.6}$$

Also, if we take $B = -1$ in (1.5), then we see that

$$p(z) = \frac{1+Az}{1-z} \qquad (-1 < A \leqq 1) \tag{1.7}$$

is analytic and univalent in $\mathbb{U}$ and the domain $p(\mathbb{U})$ is the right half-plane satisfying

$$\operatorname{Re}\big(p(z)\big) > \frac{1}{2}(1-A) \geqq 0. \tag{1.8}$$

Hence, we see that the Janowski function maps the open unit disk $\mathbb{U}$ onto some domain which is on the right half-plane.

And, as the generalization of Janowski function, Kuroki, Owa and Srivastava [2] have discussed the function

$$p(z) = \frac{1+Az}{1+Bz}$$

for some complex parameters A and B which satisfy one of following conditions

$$\begin{cases} (i)\ |B| < 1,\ A \neq B, \text{ and } \operatorname{Re}(1 - A\overline{B}) \geqq |A - B| \\ (ii)\ |B| = 1,\ A \neq B,\ |A| \leqq 1, \text{ and } 1 - A\overline{B} > 0. \end{cases}$$

First, for some complex numbers A and B which satisfy the following condition

$$(i)\quad |B| < 1,\ A \neq B, \text{ and } \operatorname{Re}(1 - A\overline{B}) \geqq |A - B|,$$

the function $p(z) = \dfrac{1+Az}{1+Bz}$ is analytic and univalent in $\mathbb{U}$ and $p(z)$ maps the open unit disk $\mathbb{U}$ onto the open disk given by

$$\left| p(z) - \frac{1 - A\overline{B}}{1 - |B|^2} \right| < \frac{|A - B|}{1 - |B|^2}.$$

Thus, it is clear that

$$\operatorname{Re}(p(z)) > \frac{\operatorname{Re}(1 - A\overline{B}) - |A - B|}{1 - |B|^2} \geqq 0 \qquad (z \in \mathbb{U}). \tag{1.9}$$

Also, for some complex numbers A and B which satisfy the following condition

$$(ii)\quad |B| = 1,\ A \neq B,\ |A| \leqq 1, \text{ and } 1 - A\overline{B} > 0,$$

the function $p(z) = \dfrac{1+Az}{1+Bz}$ is analytic and univalent in $\mathbb{U}$ and the domain $p(\mathbb{U})$ is the right half-plane satisfying

$$\operatorname{Re}(p(z)) > \frac{1 - |A|^2}{2(1 - A\overline{B})} \geqq 0. \tag{1.10}$$

Hence, we see that the generalized Janowski function maps the open unit disk $\mathbb{U}$ onto some domain which is on the right half-plane.

We define the following differential operator due to Sălăgean [8].
For a function $f(z)$ and $j = 1, 2, 3, \cdots$,

$$D^0 f(z) = f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.11}$$

$$D^1 f(z) = Df(z) = zf'(z) = z + \sum_{n=2}^{\infty} n a_n z^n, \tag{1.12}$$

$$D^j f(z) = D\big(D^{j-1} f(z)\big) = z + \sum_{n=2}^{\infty} n^j a_n z^n. \tag{1.13}$$

Also, we meditate the following integral operator

$$D^{-1} f(z) = \int_0^z \frac{f(\zeta)}{\zeta}\, d\zeta = z + \sum_{n=2}^{\infty} n^{-1} a_n z^n, \tag{1.14}$$

$$D^{-j} f(z) = D^{-1}\big(D^{-(j-1)} f(z)\big) = z + \sum_{n=2}^{\infty} n^{-j} a_n z^n \tag{1.15}$$

for any negative integers.

Then, for $f(z) \in \mathcal{A}$ given by (1.1), we know that

$$D^j f(z) = z + \sum_{n=2}^{\infty} n^j a_n z^n \quad (j = 0,\ \pm 1,\ \pm 2,\ \cdots). \tag{1.16}$$

Using the above operator $D^j f(z)$, we consider the subclass $\mathcal{S}_j^k(\alpha)$ of $\mathcal{A}$ as follows:

$$\mathcal{S}_j^k(\alpha) = \left\{ f(z) \in \mathcal{A} : \operatorname{Re}\left(\frac{D^k f(z)}{D^j f(z)} \right) > \alpha \quad (z \in \mathbb{U};\ 0 \leqq \alpha < 1) \right\}.$$

Remark 1.1 Noting

$$\frac{D^1 f(z)}{D^0 f(z)} = \frac{zf'(z)}{f(z)}, \quad \frac{D^2 f(z)}{D^1 f(z)} = \frac{z\big(zf'(z)\big)'}{zf'(z)} = 1 + \frac{zf''(z)}{f'(z)},$$

we see that

$$\mathcal{S}_0^1(\alpha) \equiv \mathcal{S}^*(\alpha), \quad \mathcal{S}_1^2(\alpha) \equiv \mathcal{K}(\alpha) \qquad (0 \leqq \alpha < 1).$$

Remark 1.2 For some α $(0 \leqq \alpha < 1)$, we find

$$\frac{D^k f(z)}{D^j f(z)} \prec \frac{1 + (1 - 2\alpha)z}{1 - z} \iff \operatorname{Re}\left(\frac{D^k f(z)}{D^j f(z)} \right) > \alpha \qquad (z \in \mathbb{U}).$$

In our investigation here, we need the following lemma concerning the differential subordination given by Miller and Mocanu [5] (see also [6, p. 132]).

Lemma 1.3 *Let the function $q(z)$ be analytic and univalent in $\mathbb{U}$. Also let $\phi(\omega)$ and $\psi(\omega)$ be analytic in a domain C containing $q(\mathbb{U})$, with*

$$\psi(\omega) \neq 0 \qquad \big(\omega \in q(\mathbb{U}) \subset C\big).$$

Set

$$Q(z) = zq'(z)\psi(q(z)) \quad and \quad h(z) = \phi(q(z)) + Q(z),$$

and suppose that

(*i*) $Q(z)$ *is starlike and univalent in* $\mathbb{U}$;

and

$$\text{(ii)} \qquad \operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) = \operatorname{Re}\left(\frac{\phi'(q(z))}{\psi(q(z))} + \frac{zQ'(z)}{Q(z)}\right) > 0 \qquad (z \in \mathbb{U}).$$

If $p(z)$ *is analytic in* $\mathbb{U}$, *with*

$$p(0) = q(0) \quad and \quad p(\mathbb{U}) \subset \mathcal{C},$$

and

$$\phi(p(z)) + zp'(z)\psi(p(z)) \prec \phi(q(z)) + zq'(z)\psi(q(z)) =: h(z) \qquad (z \in \mathbb{U}),$$

then

$$p(z) \prec q(z) \qquad (z \in \mathbb{U})$$

and $q(z)$ *is the best dominant of this subordination.*

By making use of lemma 1.3, Kuroki, Owa and Srivastava [2] have investigated some subordination criteria for the generalized Janowski functions and deduced the following lemma.

Lamma 1.4 *Let the function* $f(z) \in \mathcal{A}$ *be so chosen that* $\frac{f(z)}{z} \neq 0 \quad (z \in \mathbb{U})$. *Also, let* α $(\alpha \neq 0)$, β $(-1 \leqq \beta \leqq 1)$, *and some complex parameters* A *and* B *which satisfy one of following conditions*

(*i*) $|B| < 1$, $A \neq B$, *and* $\operatorname{Re}(1 - A\overline{B}) \geqq |A - B|$ *be so that*

$$\frac{\beta(1-\alpha)}{\alpha} + \frac{(1+\beta)\{\operatorname{Re}(1 - A\overline{B}) - |A - B|\}}{1 - |B|^2} + \frac{1-\beta}{1+|A|} + \frac{1+\beta}{1+|B|} - 1 \geqq 0,$$

(*ii*) $|B| = 1$, $A \neq B$, $|A| \leqq 1$, *and* $1 - A\overline{B} > 0$ *be so that*

$$\frac{\beta(1-\alpha)}{\alpha} + \frac{(1+\beta)(1-|A|^2)}{2(1-A\overline{B})} + \frac{(1-\beta)(1-|A|)}{2(1+|A|)} \geqq 0.$$

If

$$\text{(1.17)} \qquad \left(\frac{zf'(z)}{f(z)}\right)^{\beta}\left(1 + \alpha\frac{zf''(z)}{f'(z)}\right) \prec h(z) \qquad (z \in \mathbb{U}),$$

where

$$h(z) = \left(\frac{1+Az}{1+Bz}\right)^{\beta-1}\left\{(1-\alpha)\frac{1+Az}{1+Bz} + \frac{\alpha(1+Az)^2 + \alpha(A-B)z}{(1+Bz)^2}\right\},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}).$$

2 Subordinations for the class defined by Sălăgean operator

Applying Sălăgean operator for $f(z) \in \mathcal{A}$, we deduced the following subordination criterion for the generalized Janowski function.

Theorem 2.1 *Let the function $f(z) \in \mathcal{A}$ be so chosen that $\frac{D^j f(z)}{z} \neq 0 \quad (z \in \mathbb{U})$. Also, let α $(\alpha \neq 0)$, β $(-1 \leqq \beta \leqq 1)$, and some complex parameters A and B which satisfy one of following conditions*

(*i*) $|B| < 1$, $A \neq B$, *and* $\mathrm{Re}(1 - A\overline{B}) \geqq |A - B|$ *be so that*

$$\frac{\beta(1-\alpha)}{\alpha} + \frac{(1+\beta)\{\mathrm{Re}(1-A\overline{B}) - |A-B|\}}{1-|B|^2} + \frac{1-\beta}{1+|A|} + \frac{1+\beta}{1+|B|} - 1 \geqq 0,$$

(*ii*) $|B| = 1$, $A \neq B$, $|A| \leqq 1$, *and* $1 - A\overline{B} > 0$ *be so that*

$$\frac{\beta(1-\alpha)}{\alpha} + \frac{(1+\beta)(1-|A|^2)}{2(1-A\overline{B})} + \frac{(1-\beta)(1-|A|)}{2(1+|A|)} \geqq 0.$$

If

$$\left(\frac{D^k f(z)}{D^j f(z)}\right)^{\beta} \left\{(1-\alpha) + \alpha\left(\frac{D^k f(z)}{D^j f(z)} + \frac{D^{k+1} f(z)}{D^k f(z)} - \frac{D^{j+1} f(z)}{D^j f(z)}\right)\right\} \prec h(z), \tag{2.1}$$

where

$$h(z) = \left(\frac{1+Az}{1+Bz}\right)^{\beta-1} \left\{(1-\alpha)\frac{1+Az}{1+Bz} + \frac{\alpha(1+Az)^2 + \alpha(A-B)z}{(1+Bz)^2}\right\},$$

then

$$\frac{D^k f(z)}{D^j f(z)} \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}).$$

Proof. If we define the function $p(z)$ by

$$p(z) = \frac{D^k f(z)}{D^j f(z)} \qquad (z \in \mathbb{U}),$$

then $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$. Further, since

$$zp'(z) = \left(\frac{D^k f(z)}{D^j f(z)}\right)\left(\frac{D^{k+1} f(z)}{D^k f(z)} - \frac{D^{j+1} f(z)}{D^j f(z)}\right),$$

the condition (2.1) can be written as follows:

$$\{p(z)\}^{\beta}\{(1-\alpha) + \alpha p(z)\} + \alpha z p'(z)\{p(z)\}^{\beta-1} \prec h(z) \qquad (z \in \mathbb{U}).$$

We also set

$$q(z) = \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}),$$

and

$$\phi(\omega) = \omega^{\beta}(1-\alpha+\alpha\omega), \quad \text{and} \quad \psi(\omega) = \alpha\omega^{\beta-1}$$

for $\omega \in q(\mathbb{U})$. Then, it is clear that the function $q(z)$ is analytic and univalent in $\mathbb{U}$ and have a positive real part in $\mathbb{U}$ for the conditions (i) and (ii).
Therefore, ϕ and ψ are analytic in a domain $\mathcal{C}$ containing $q(\mathbb{U})$, with

$$\psi(\omega) = \alpha\omega^{\beta-1} \neq 0 \qquad (\omega \in q(\mathbb{U}) \subset \mathcal{C}).$$

Also, for the function $Q(z)$ given by

$$Q(z) = zq'(z)\psi(q(z)) = \frac{\alpha(A-B)z(1+Az)^{\beta-1}}{(1+Bz)^{\beta+1}},$$

we obtain

$$\frac{zQ'(z)}{Q(z)} = \frac{1-\beta}{1+Az} + \frac{1+\beta}{1+Bz} - 1. \tag{2.2}$$

Furthermore, we have

$$\begin{aligned} h(z) &= \phi(q(z)) + Q(z) \\ &= \left(\frac{1+Az}{1+Bz}\right)^{\beta}\left(1-\alpha+\alpha\frac{1+Az}{1+Bz}\right) + \frac{\alpha(A-B)z(1+Az)^{\beta-1}}{(1+Bz)^{\beta+1}} \end{aligned}$$

and

$$\frac{zh'(z)}{Q(z)} = \frac{\beta(1-\alpha)}{\alpha} + (1+\beta)q(z) + \frac{zQ'(z)}{Q(z)}. \tag{2.3}$$

Hence,
(i) For the complex numbers A and B such that

$$|B| < 1, \ A \neq B, \ \text{and} \ \operatorname{Re}(1-A\overline{B}) \geqq |A-B|,$$

it follows from (2.2) and (2.3) that

$$\operatorname{Re}\left(\frac{zQ'(z)}{Q(z)}\right) > \frac{1-\beta}{1+|A|} + \frac{1+\beta}{1+|B|} - 1 \geqq 0,$$

and

$$\begin{aligned} \operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) > \frac{\beta(1-\alpha)}{\alpha} &+ \frac{(1+\beta)\{\operatorname{Re}(1-A\overline{B}) - |A-B|\}}{1-|B|^2} \\ &+ \frac{1-\beta}{1+|A|} + \frac{1+\beta}{1+|B|} - 1 \geqq 0 \qquad (z \in \mathbb{U}). \end{aligned}$$

(*ii*) For the complex numbers A and B such that

$$|B| = 1, \ |A| \leqq 1, \ A \neq B, \ \text{and} \ \ 1 - A\overline{B} > 0,$$

from (2.2) and (2.3), we get

$$\operatorname{Re}\left(\frac{zQ'(z)}{Q(z)}\right) > \frac{1-\beta}{1+|A|} + \frac{1}{2}(1+\beta) - 1 = \frac{(1-\beta)(1-|A|)}{2(1+|A|)} \geqq 0,$$

and

$$\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) > \frac{\beta(1-\alpha)}{\alpha} + \frac{(1+\beta)(1-|A|^2)}{2(1-A\overline{B})} + \frac{(1-\beta)(1-|A|)}{2(1+|A|)} \geqq 0 \qquad (z \in \mathbb{U}).$$

Since all conditions of Lemma 1.3 are satisfied, we conclude that

$$\frac{D^k f(z)}{D^j f(z)} \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}),$$

which completes the proof of Theorem 2.1. □

Letting $k = j + 1$ in Theorem 2.1, we obtain

Corollary 2.2 *Let the function $f(z) \in \mathcal{A}$ be so chosen that $\frac{D^j f(z)}{z} \neq 0 \ \ (z \in \mathbb{U})$. Also, let α $(\alpha \neq 0)$, β $(-1 \leqq \beta \leqq 1)$, and some complex parameters A and B which satisfy one of following conditions*

(*i*) $|B| < 1$, $A \neq B$, *and* $\operatorname{Re}(1 - A\overline{B}) \geqq |A - B|$ *be so that*

$$\frac{\beta(1-\alpha)}{\alpha} + \frac{(1+\beta)\{\operatorname{Re}(1-A\overline{B}) - |A-B|\}}{1-|B|^2} + \frac{1-\beta}{1+|A|} + \frac{1+\beta}{1+|B|} - 1 \geqq 0,$$

(*ii*) $|B| = 1$, $A \neq B$, $|A| \leqq 1$, *and* $1 - A\overline{B} > 0$ *be so that*

$$\frac{\beta(1-\alpha)}{\alpha} + \frac{(1+\beta)(1-|A|^2)}{2(1-A\overline{B})} + \frac{(1-\beta)(1-|A|)}{2(1+|A|)} \geqq 0.$$

If

$$\left(\frac{D^{j+1}f(z)}{D^j f(z)}\right)^{\beta}\left(1 - \alpha + \alpha\frac{D^{j+2}f(z)}{D^{j+1}f(z)}\right) \prec h(z), \tag{2.2}$$

where

$$h(z) = \left(\frac{1+Az}{1+Bz}\right)^{\beta-1}\left\{(1-\alpha)\frac{1+Az}{1+Bz} + \frac{\alpha(1+Az)^2 + \alpha(A-B)z}{(1+Bz)^2}\right\},$$

then

$$\frac{D^{j+1}f(z)}{D^j f(z)} \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}).$$

Remark 2.3 Setting $j = 0$ in Corollary 2.2, we obtain Lamma 1.4 proven by Kuroki, Owa and Srivastava [2].

Also, if we assume that $\alpha = 1$, $\beta = A = 0$, and $B = \dfrac{1-\mu}{1+\mu}e^{i\theta}$ $(0 \leqq \mu < 1,\ 0 \leqq \theta < 2\pi)$, Corollary 2.2 becomes the following corollary.

Corollary 2.4 *If* $f(z) \in \mathcal{A}$ $\left(\dfrac{D^j f(z)}{z} \neq 0 \text{ in } \mathbb{U}\right)$ *satisfies*

$$\frac{D^{j+2}f(z)}{D^{j+1}f(z)} \prec \frac{1+\mu-(1-\mu)e^{i\theta}z}{1+\mu+(1-\mu)e^{i\theta}z} \qquad (z \in \mathbb{U};\ 0 \leqq \theta < 2\pi)$$

for some μ $(0 \leqq \mu < 1)$, *then*

$$\frac{D^{j+1}f(z)}{D^j f(z)} \prec \frac{1+\mu}{1+\mu+(1-\mu)e^{i\theta}z} \qquad (z \in \mathbb{U}).$$

From the above corollary, we have

$$\mathrm{Re}\left(\frac{D^{j+2}f(z)}{D^{j+1}f(z)}\right) > \mu \quad \Longrightarrow \quad \mathrm{Re}\left(\frac{D^{j+1}f(z)}{D^j f(z)}\right) > \frac{1+\mu}{2} \qquad (z \in \mathbb{U};\ 0 \leqq \mu < 1).$$

Thus, we see that

$$\begin{aligned} f(z) \in \mathcal{S}_{j+1}^{j+2}(\mu) \quad &\Longrightarrow \quad f(z) \in \mathcal{S}_j^{j+1}\left(\frac{1+\mu}{2}\right) \quad \Longrightarrow \quad f(z) \in \mathcal{S}_{j-1}^{j}\left(\frac{3+\mu}{4}\right) \\ &\Longrightarrow \quad \cdots \quad \Longrightarrow \quad f(z) \in \mathcal{S}_1^2\left(\frac{2^j-1+\mu}{2^j}\right) \\ &\Longrightarrow \quad f(z) \in \mathcal{S}_0^1\left(\frac{2^{j+1}-1+\mu}{2^{j+1}}\right) \qquad (z \in \mathbb{U};\ 0 \leqq \mu < 1). \end{aligned}$$

In particular, we find

$$\begin{aligned} f(z) \in \mathcal{S}_{j+1}^{j+2}(\mu) \quad &\Longrightarrow \quad f(z) \in \mathcal{K}\left(\frac{2^j-1+\mu}{2^j}\right) \\ &\Longrightarrow \quad f(z) \in \mathcal{S}^*\left(\frac{2^{j+1}-1+\mu}{2^{j+1}}\right) \qquad (z \in \mathbb{U};\ 0 \leqq \mu < 1). \end{aligned}$$

And, taking $j = 0$ and $\mu = 0$, we find the fact that every convex function is starlike of order $\dfrac{1}{2}$. This fact is well-known as the Marx-Strohhäcker theorem in Univalent Function Theory (cf. [4], [9]).

3 Subordination criteria for other analytic functions

In this section, by making use of Lemma 1.3, we consider some subordination criteria concerning analytic function $\dfrac{D^j f(z)}{z}$ for $f(z) \in \mathcal{A}$.

Theorem 3.1 *Let α $(\alpha \neq 0)$, β $(-1 \leqq \beta \leqq 1)$, and some complex parameters A and B which satisfy one of following conditions*

(*i*) $|B| < 1$, $A \neq B$, *and* $\mathrm{Re}(1 - A\overline{B}) \geqq |A - B|$ *be so that*

$$\frac{\beta}{\alpha} + \frac{1-\beta}{1+|A|} + \frac{1+\beta}{1+|B|} - 1 \geqq 0,$$

(*ii*) $|B| = 1$, $A \neq B$, $|A| \leqq 1$, *and* $1 - A\overline{B} > 0$ *be so that*

$$\frac{\beta}{\alpha} + \frac{(1-\beta)(1-|A|)}{2(1+|A|)} \geqq 0.$$

If $f(z) \in \mathcal{A}$ *satisfies*

$$\left(\frac{D^j f(z)}{z}\right)^{\beta}\left(1 - \alpha + \alpha\frac{D^{j+1} f(z)}{D^j f(z)}\right) \prec \left(\frac{1+Az}{1+Bz}\right)^{\beta} + \frac{\alpha(A-B)z(1+Az)^{\beta-1}}{(1+Bz)^{\beta+1}}, \tag{3.1}$$

then

$$\frac{D^j f(z)}{z} \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}).$$

Proof. If we define the function $p(z)$ by

$$p(z) = \frac{D^j f(z)}{z} \qquad (z \in \mathbb{U}),$$

then $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$ and the condition (3.1) can be written as follows:

$$\{p(z)\}^{\beta} + \alpha z p'(z)\{p(z)\}^{\beta-1} \prec h(z) \qquad (z \in \mathbb{U}).$$

We also set

$$q(z) = \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}),$$

and

$$\phi(\omega) = \omega^{\beta}, \quad \text{and} \quad \psi(\omega) = \alpha\omega^{\beta-1}$$

for $\omega \in q(\mathbb{U})$. Then, the function $q(z)$ is analytic and univalent in $\mathbb{U}$ and satisfies

$$\mathrm{Re}(q(z)) > 0 \qquad (z \in \mathbb{U})$$

for the condition (i) and (ii).
Thus, the functions ϕ and ψ satisfy the conditions required by Lemma 1.3.
Further, for the functions $Q(z)$ and $h(z)$ given by

$$Q(z) = zq'(z)\psi(q(z)) \quad \text{and} \quad h(z) = \phi(q(z)) + Q(z),$$

we have

$$\frac{zQ'(z)}{Q(z)} = \frac{1-\beta}{1+Az} + \frac{1+\beta}{1+Bz} - 1 \quad \text{and} \quad \frac{zh'(z)}{Q(z)} = \frac{\beta}{\alpha} + \frac{zQ'(z)}{Q(z)}.$$

Then, similarly to proof of Theorem 2.1, we see that

$$\operatorname{Re}\left(\frac{zQ'(z)}{Q(z)}\right) > 0 \quad \text{and} \quad \operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) > 0 \qquad (z \in \mathbb{U})$$

for the conditions (i) and (ii).

Thus, by applying Lemma 1.3, we conclude that $p(z) \prec q(z) \quad (z \in \mathbb{U})$.
The proof of the theorem is completed. □

In Theorem 3.1, taking $\alpha = 1$, $\beta = A = 0$, and $B = \frac{1-\nu}{\nu}e^{i\theta} \quad \left(\frac{1}{2} \leqq \nu < 1,\ 0 \leqq \theta < 2\pi\right)$, we obtain the following corollary.

Corollary 3.2 *If $f(z) \in \mathcal{A}$ satisfies*

$$\frac{D^{j+1}f(z)}{D^j f(z)} \prec \frac{\nu}{\nu + (1-\nu)e^{i\theta}z} \qquad (z \in \mathbb{U};\ 0 \leqq \theta < 2\pi)$$

for some ν $\left(\frac{1}{2} \leqq \nu < 1\right)$, then

$$\frac{D^j f(z)}{z} \prec \frac{\nu}{\nu + (1-\nu)e^{i\theta}z} \qquad (z \in \mathbb{U}).$$

Also, making $\alpha = \beta = 1$, $A = 0$, and $B = \frac{1-\nu}{\nu}e^{i\theta} \quad \left(\frac{1}{2} \leqq \nu < 1,\ 0 \leqq \theta < 2\pi\right)$ in Theorem 3.1, we get

Corollary 3.3 *If $f(z) \in \mathcal{A}$ satisfies*

$$\frac{D^{j+1}f(z)}{z} \prec \left(\frac{\nu}{\nu + (1-\nu)e^{i\theta}z}\right)^2 \qquad (z \in \mathbb{U};\ 0 \leqq \theta < 2\pi)$$

for some ν $\left(\frac{1}{2} \leqq \nu < 1\right)$, then

$$\frac{D^j f(z)}{z} \prec \frac{\nu}{\nu + (1-\nu)e^{i\theta}z} \qquad (z \in \mathbb{U}).$$

The above corollaries derive each of the facts that

$$\operatorname{Re}\left(\frac{D^{j+1}f(z)}{D^j f(z)}\right) > \nu \quad \Longrightarrow \quad \operatorname{Re}\left(\frac{D^j f(z)}{z}\right) > \nu \qquad \left(z \in \mathbb{U};\ \frac{1}{2} \leqq \nu < 1\right),$$

and

$$\operatorname{Re}\sqrt{\frac{D^{j+1}f(z)}{z}} > \nu \quad \Longrightarrow \quad \operatorname{Re}\left(\frac{D^j f(z)}{z}\right) > \nu \qquad \left(z \in \mathbb{U};\ \frac{1}{2} \leqq \nu < 1\right).$$

In particular, for $j = 0$, we see that

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \nu \quad \Longrightarrow \quad \mathrm{Re}\left(\frac{f(z)}{z}\right) > \nu \qquad \left(z \in \mathbb{U};\ \frac{1}{2} \leqq \nu < 1\right),$$

and

$$\mathrm{Re}\sqrt{f'(z)} > \nu \quad \Longrightarrow \quad \mathrm{Re}\left(\frac{f(z)}{z}\right) > \nu \qquad \left(z \in \mathbb{U};\ \frac{1}{2} \leqq \nu < 1\right).$$

Here, taking $\nu = \frac{1}{2}$, we find some results well-known as the Marx-Strohhäcker theorem in Univalent Function Theory (cf. [4], [9]).

Also, letting $j = 1$ in Corollary 3.2, we get the following fact:

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \nu \quad \Longrightarrow \quad \mathrm{Re}\,(f'(z)) > \nu \qquad \left(z \in \mathbb{U};\ \frac{1}{2} \leqq \nu < 1\right).$$

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